

# Cycle Double Covers and Semi-Kotzig Frame

Dong Ye and Cun-Quan Zhang

Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310

Emails: dye@math.wvu.edu; cqzhang@math.wvu.edu

January 25, 2013

## Abstract

Let  $H$  be a cubic graph admitting a 3-edge-coloring  $c : E(H) \rightarrow \mathbb{Z}_3$  such that the edges colored by 0 and  $\mu \in \{1, 2\}$  induce a Hamilton circuit of  $H$  and the edges colored by 1 and 2 induce a 2-factor  $F$ . The graph  $H$  is semi-Kotzig if switching colors of edges in any even subgraph of  $F$  yields a new 3-edge-coloring of  $H$  having the same property as  $c$ . A spanning subgraph  $H$  of a cubic graph  $G$  is called a *semi-Kotzig frame* if the contracted graph  $G/H$  is even and every non-circuit component of  $H$  is a subdivision of a semi-Kotzig graph.

In this paper, we show that a cubic graph  $G$  has a circuit double cover if it has a semi-Kotzig frame with at most one non-circuit component. Our result generalizes some results of Goddyn (1988), and Häggkvist and Markström [J. Combin. Theory Ser. B (2006)].

## 1 Introduction

Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . A *circuit* of  $G$  is a connected 2-regular subgraph. A subgraph of  $G$  is *even* if every vertex is of even degree. An even subgraph of  $G$  is also called a *cycle* in the literatures dealing with cycle covers of graphs [13] [12] [20]. Every even graph has a circuit decomposition. A set  $\mathcal{C}$  of even-subgraphs of  $G$  is an *even-subgraph double cover* (cycle double cover) if each edge of  $G$  is contained by precisely two even-subgraphs in  $\mathcal{C}$ . The Circuit Double Cover Conjecture was made independently by Szekeres [17] and Seymour [16].

**Conjecture 1.1** (Szekeres [17] and Seymour [16]). *Every bridgeless graph  $G$  has a circuit double cover.*

It suffices to show that the Circuit Double Cover Conjecture holds for bridgeless cubic graphs [13]. The Circuit Double Cover Conjecture has been verified for several classes of graphs; for example, cubic graphs with Hamilton paths [18] (also see [4]), cubic graphs with oddness two [9] and four [8, 7], Petersen-minor-free graphs [1].

A cubic graph  $H$  is a *spanning minor* of a cubic graph  $G$  if some subdivision of  $H$  is a spanning subgraph of  $G$ . In [3], Goddyn showed that a cubic graph  $G$  has a circuit double cover if it contains the Petersen graph as a spanning minor. Goddyn's result is further improved by Häggkvist and Markström [6] who showed that a cubic graph  $G$  has a circuit double cover if it contains a 2-connected simple cubic graph with no more than 10 vertices as a spanning minor.

A *Kotzig graph* is a cubic graph  $H$  with a 3-edge-coloring  $c : E(G) \rightarrow \mathbb{Z}_3$  such that  $c^{-1}(\alpha) \cup c^{-1}(\beta)$  induces a Hamilton circuit of  $G$  for every pair  $\alpha, \beta \in \mathbb{Z}_3$ . The family of all Kotzig graphs is denoted by  $\mathcal{K}$ .

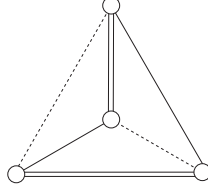


Figure 1: The Kotzig graph  $K_4$ .

**Theorem 1.2** (Goddyn [3], Häggkvist and Markström [5]). *If a cubic graph  $G$  contains a Kotzig graph as a spanning minor, then  $G$  has a 6-even-subgraph double cover.*

By Theorem 1.2, any cubic graph  $G$  containing some member of  $\mathcal{K}$  as a spanning minor has a circuit double cover. However, we do not know yet whether every 3-connected cubic graph contains a member of  $\mathcal{K}$  as a spanning minor (Conjecture 1.3).

According to their observations [5, 6], Häggkvist and Markström proposed the following conjectures.

**Conjecture 1.3** (Häggkvist and Markström, [5]). *Every 3-connected cubic graph contains a Kotzig graph as a spanning minor<sup>1</sup>.*

Häggkvist and Markström [5] proposed another conjecture (Conjecture 2.3) in a more general form. We will discuss this conjecture in the last section (Remark).

One of approaches to the CDC conjecture is to find a sup-family  $\mathcal{X}$  of  $\mathcal{K}$  such that every bridgeless cubic graph containing a member of  $\mathcal{X}$  as a spanning minor has a CDC. Following this direction of approach, Goddyn [3], Häggkvist and Markström [5] introduce some sup-families of  $\mathcal{K}$ , named iterated-Kotzig graphs, switchable-CDC graphs and semi-Kotzig graphs. They will be defined in next subsections and their relations are shown in Figure 2.

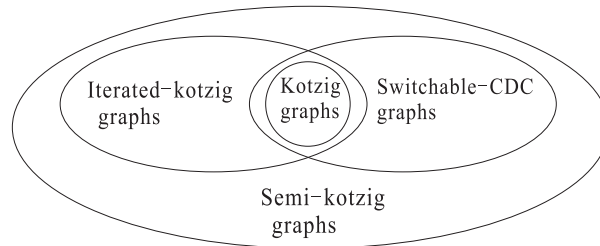


Figure 2: The inclusion relations for these four families: Kotzig graphs, iterated-Kotzig graphs, switchable CDC graphs, Semi-Kotzig graphs.

<sup>1</sup> It is pointed out in [10] that the 3-edge-connectivity is not enough for the existence of such spanning minor, and he suggested that an extra requirement of cyclical 4-edge-connectivity is necessary.

## Iterated-Kotzig graphs

**Definition 1.4.** An *iterated-Kotzig graph*  $H$  is a cubic graph constructed as following [5]: Let  $\mathcal{K}_0$  be a set of Kotzig graphs with a 3-edge-coloring  $c : E(G) \rightarrow \mathbb{Z}_3$ ; A cubic graph  $H \in \mathcal{K}_{i+1}$  can be constructed from a graph  $H_i \in \mathcal{K}_i$  and a graph  $H_0 \in \mathcal{K}_0$  by deleting one edge colored by 0 from each of them and joining the two vertices of degree two in  $H_0$  to the two vertices of degree two in  $H_i$ , respectively (the two new edges will be colored by 0).

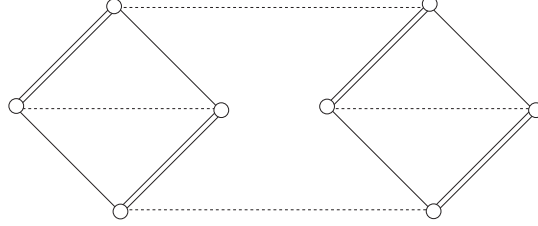


Figure 3: An iterated-Kotzig graph generated from two  $K_4$ 's.

**Theorem 1.5** (Häggkvist and Markström, [5]). *If a cubic graph  $G$  contains an iterated Kotzig graph as a spanning minor, then  $G$  has a 6-even-subgraph double cover.*

## Semi-Kotzig graphs and switchable-CDC graphs

**Definition 1.6.** Let  $G$  be a cubic graph with a 3-edge-coloring  $c : E(G) \rightarrow \mathbb{Z}_3$  and the following property

- (\*) edges in colors 0 and  $\mu$  ( $\mu \in \{1, 2\}$ ) induce a Hamilton circuit.

Let  $F$  be the even 2-factor induced by edges in colors 1 and 2. If, for every even subgraph  $S \subseteq F$ , switching colors 1 and 2 of the edges of  $S$  yields a new 3-edge-coloring having the property (\*), then each of these  $2^{t-1}$  3-edge-coloring is called a *semi-Kotzig coloring* where  $t$  is the number of components of  $F$ . A cubic graph  $G$  with a semi-Kotzig coloring is called a *semi-Kotzig graph*. If  $F$  has at most two components ( $t \leq 2$ ), then  $G$  is said to be a *switchable-CDC graph* (defined in [5]).

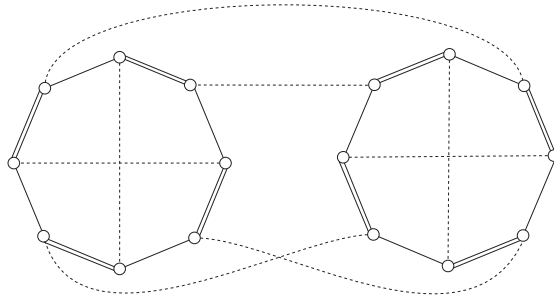


Figure 4: A semi-Kotzig graph.

**Theorem 1.7** (Häggkvist and Markström, [5]). *If a cubic graph  $G$  contains a switchable-CDC graph as a spanning minor, then  $G$  has a 6-even-subgraph double cover.*

An iterated-Kotzig graph has a semi-Kotzig coloring and hence is a semi-Kotzig graph. But a semi-Kotzig graph is not necessary an iterated-Kotzig graph. For example, the semi-Kotzig graph in Figure 4 is not an iterated-Kotzig graph. Hence we have the following relations (also see Figure 2)

$$\text{Kotzig} \subset \text{Iterated-Kotzig} \subset \text{Semi-Kotzig}; \quad (1)$$

$$\text{Kotzig} \subset \text{Switchable-CDC} \subset \text{Semi-Kotzig}. \quad (2)$$

The following theorem was announced in [3] with an outline of proof.

**Theorem 1.8** (Goddyn, [3]). *If a cubic graph  $G$  contains a semi-Kotzig graph as a spanning minor, then  $G$  has a 6-even-subgraph double cover.*

The main theorem (Theorem 1.17) of the paper strengthens all those early results (Theorems 1.2, 1.5, 1.7 and 1.8).

## Kotzig frame, semi-Kotzig frame

A 2-factor  $F$  of a cubic graph is *even* if every component of  $F$  is of even length. If a cubic graph  $G$  has an even 2-factor, then the graph  $G$  has many nice properties:  $G$  is 3-edge-colorable,  $G$  has a circuit double cover and strong circuit double cover, etc.

The following concepts were introduced in [5] as a generalization of even 2-factors.

**Definition 1.9.** Let  $G$  be a cubic graph. A spanning subgraph  $H$  of  $G$  is called a *frame* of  $G$  if the contracted graph  $G/H$  is an even graph.

For a subgraph  $H$  of  $G$ , the *suppressed graph*  $\overline{H}$  of  $H$  is the graph obtained from  $H$  by suppressing all degree 2 vertices.

**Definition 1.10.** Let  $G$  be a cubic graph. A frame  $H$  of  $G$  is called a *Kotzig frame* (or *iterated-Kotzig frame*, or *switchable-CDC frame*, or *semi-Kotzig frame*) of  $G$  if, for each non-circuit component  $H_i$  of  $H$ , the suppressed graph  $\overline{H}_i$  is a Kotzig graph (or an iterated-Kotzig graph, or a switchable-CDC graph, or a semi-Kotzig graph, respectively).

Similar to the relations described in (1) and (2), we have the same relations between those frames:

$$\text{Kotzig frame} \subset \text{Iterated-Kotzig frame} \subset \text{semi-Kotzig frame};$$

$$\text{Kotzig frame} \subset \text{Switchable-CDC frame} \subset \text{semi-Kotzig frame}.$$

**Theorem 1.11** (Häggkvist and Markström, [5]). *Let  $G$  be a bridgeless cubic graph  $G$ . If  $G$  contains a Kotzig frame with at most one non-circuit component, then  $G$  has a 6-even-subgraph double cover.*

According to their observations, they further make the following conjecture.

**Conjecture 1.12** (Häggkvist and Markström, [5]). *Every bridgeless cubic graph with a Kotzig frame has a 6-even-subgraph double cover.*

The following theorem provides a partial solution to Conjecture 1.12.

**Theorem 1.13** (Zhang and Zhang, [21]). *Let  $G$  be a bridgeless cubic graph. If  $G$  contains a Kotzig frame  $H$  such that  $G/H$  is a tree if parallel edges are identified as a single edge, then  $G$  has a 6-even-subgraph double cover.*

We conjecture that the result in Conjecture 1.12 still holds if a Kotzig frame is replaced by a semi-Kotzig frame.

**Conjecture 1.14.** *Every bridgeless cubic graph with a semi-Kotzig frame has a 6-even-subgraph double cover.*

Häggkvist and Markström showed Conjecture 1.14 holds for iterated-kotzig frames and switchable-CDC frames with at most one non-circuit component.

**Theorem 1.15** (Häggkvist and Markström, [5]). *Let  $G$  be a bridgeless cubic graph  $G$ . If  $G$  contains an iterated-Kotzig frame with at most one non-circuit component, then  $G$  has a 6-even-subgraph double cover.*

**Theorem 1.16** (Häggkvist and Markström, [5]). *Let  $G$  be a bridgeless cubic graph  $G$ . If  $G$  contains a switchable-CDC frame with at most one non-circuit component, then  $G$  has a 6-even-subgraph double cover.*

The following theorem is the main result of the paper, which verifies that Conjecture 1.14 holds if a semi-Kotzig frame has at most one non-circuit component. Since Kotzig graphs, iterated-Kotzig graphs are semi-Kotzig graphs but not vice versa, Theorems 1.2, 1.5, 1.7, 1.8, 1.11, 1.15 and 1.16 are corollaries of our result. The proof of the theorem will be given in Section 2.

**Theorem 1.17.** *Let  $G$  be a bridgeless cubic graph. If  $G$  contains a semi-Kotzig frame  $H$  with at most one non-circuit component, then  $G$  has a 6-even-subgraph double cover.*

## 2 Proof of Theorem 1.17

The following well-known fact will be applied in the proof of the main theorem (Theorem 1.17).

**Lemma 2.1.** *If a cubic graph has an even 2-factor  $F$ , then  $G$  has a 3-even-subgraph double cover  $\mathcal{C}$  such that  $F \in \mathcal{C}$ .*

**Definition 2.2.** Let  $H$  be a bridgeless subgraph of a cubic graph  $G$ . A mapping  $c : E(H) \rightarrow \mathbb{Z}_3$  is called a *parity 3-edge-coloring* of  $H$  if, for each vertex  $v \in H$  and each  $\mu \in \mathbb{Z}_3$ ,

$$|c^{-1}(\mu) \cap E(v)| \equiv |E(v) \cap E(H)| \pmod{2}.$$

It is obvious that if  $H$  itself is cubic, then a parity 3-edge-coloring is a proper 3-edge-coloring (traditional definition).

**Preparation of the proof.** Let  $H_0$  be the component of  $H$  such that  $H_0$  is a subdivision of a semi-Kotzig graph and each  $H_i$ ,  $1 \leq i \leq t$ , be a circuit component of  $H$  of even length. Let  $M = E(G) - E(H)$ , and  $H^* = H - H_0$ .

Given an initial semi-Kotzig coloring  $c_0 : E(\overline{H}_0) \rightarrow \mathbb{Z}_3$  of  $\overline{H}_0$ , then  $F_0 = c_0^{-1}(1) \cup c_0^{-1}(2)$  is a 2-factor of  $\overline{H}_0$  and  $c_0^{-1}(0) \cup c_0^{-1}(\mu)$  is a Hamilton circuit of  $\overline{H}_0$  for each  $\mu \in \{1, 2\}$ .

The semi-Kotzig coloring  $c_0$  of  $\overline{H}_0$  can be considered as an edge-coloring of  $H_0$ : each induced path is colored with the same color as its corresponding edge in  $\overline{H}_0$  (note, this edge-coloring of  $H_0$  is a parity 3-edge-coloring, which may not be a proper 3-edge-coloring).

The strategy of the proof is to show that  $G$  can be covered by three subgraphs  $G(0, 1)$ ,  $G(0, 2)$  and  $G(1, 2)$  such that each  $G(\alpha, \beta)$  has a 2-even-subgraph cover which covers the edges of  $M \cap E(G(\alpha, \beta))$  twice and the edges of  $E(H) \cap E(G(\alpha, \beta))$  once. In order to prove this, we are going to show that the three subgraphs  $G(\alpha, \beta)$  have the following properties:

- (i) the suppressed cubic graph  $\overline{G(\alpha, \beta)}$  is 3-edge-colorable (so that Lemma 2.1 can be applied to each of them);
- (ii)  $c_0^{-1}(\alpha) \cup c_0^{-1}(\beta) \subseteq G(\alpha, \beta)$  for each pair  $\alpha, \beta \in \mathbb{Z}_3$ ;
- (iii) The even subgraph  $H^*$  has a decomposition,  $H_1^*$  and  $H_2^*$ , each of which is an even subgraph, (here, for technical reason, let  $H_0^* = \emptyset$ ), such that  $H_\alpha^* \cup H_\beta^* \subseteq G(\alpha, \beta)$ , for each  $\{\alpha, \beta\} \subset \mathbb{Z}_3$ ;
- (iv) each  $e \in M = E(G) - E(H)$  is contained in precisely one member of  $\{G(0, 1), G(0, 2), G(1, 2)\}$ ;
- (v) and most important, the subgraph  $c^{-1}(\alpha) \cup c^{-1}(\beta) \cup H_\alpha^* \cup H_\beta^*$  in  $G(\alpha, \beta)$  corresponds to an even 2-factor of  $\overline{G(\alpha, \beta)}$ .

Can we decompose  $H^*$  and find a partition of  $M = E(G) - E(H)$  to satisfy (v)? One may also notice that the initial semi-Kotzig coloring  $c$  may not be appropriate. However, the color-switchability of the semi-Kotzig component  $H_0$  may help us to achieve the goal. The properties described above in the strategy will be proved in the following claim.

We claim that  $G$  has the following property:

- (\*) *There is a semi-Kotzig coloring  $c_0$  of  $\overline{H}_0$ , a decomposition  $\{H_1^*, H_2^*\}$  of  $H^*$  and a partition  $\{N_{(0,1)}, N_{(0,2)}, N_{(1,2)}\}$  of  $M$  such that, let  $C_{(\alpha,\beta)} = c_0^{-1}(\alpha) \cup c_0^{-1}(\beta)$ ,*
  - (1) *for each  $\mu \in \{1, 2\}$ ,  $C_{(0,\mu)} \cup H_\mu^*$  corresponds to an even 2-factor of  $\overline{G(0, \mu)} = \overline{G[C_{(0,\mu)} \cup H_\mu^* \cup N_{(0,\mu)}]}$ , and*
  - (2)  *$C_{(1,2)} \cup H^*$  corresponds to an even 2-factor of  $\overline{G(1, 2)} = \overline{G[C_{(1,2)} \cup H^* \cup N_{(1,2)}]}$ .*

**Proof of (\*).** Let  $G$  be a minimum counterexample to (\*). Let  $c : E(H) \rightarrow \mathbb{Z}_3$  be a parity 3-edge-coloring of  $H$  such that

- (1) the restriction of  $c$  on  $\overline{H}_0$  is a semi-Kotzig coloring, and
- (2)  $E(H^*) \subseteq c^{-1}(1) \cup c^{-1}(2)$  (a set of mono-colored circuits).

Let

$$F = c^{-1}(1) \cup c^{-1}(2) = E(H) - c^{-1}(0).$$

Partition the matching  $M$  as follows. For each edge  $e = xy \in M$ ,  $xy \in M_{(\alpha,\beta)}$  ( $\alpha \leq \beta$  and  $\alpha, \beta \in \mathbb{Z}_3$ ) if  $x$  is incident with two  $\alpha$ -colored edges and  $y$  is incident with two  $\beta$ -colored edges. So,

the matching  $M$  is partitioned into six subsets:

$$M_{(0,0)}, M_{(0,1)}, M_{(0,2)}, M_{(1,1)}, M_{(1,2)} \text{ and } M_{(2,2)}.$$

Note that this partition will be adjusted whenever the parity 3-edge-coloring is adjusted.

**Claim 1.**  $M_{(0,\mu)} \cap G[V(H_0)] = \emptyset$ , for each  $\mu \in \mathbb{Z}_3$ .

Suppose that  $e = xy \in M_{(0,\mu)}$  where  $x$  is incident with two 0-colored edges of  $H_0$ . Then, in the graph  $\overline{G - e}$ , the spanning subgraph  $H$  retains the same property as itself in  $G$ . Since  $\overline{G - e}$  is smaller than  $G$ ,  $\overline{G - e}$  satisfies (\*):  $\overline{H_0}$  has a semi-Kotzig coloring  $c_0$  and  $M - e$  has a partition  $\{N_{(0,1)}, N_{(0,2)}, N_{(1,2)}\}$  and  $H^*$  has a decomposition  $\{H_1^*, H_2^*\}$ . In the semi-Kotzig coloring  $c_0$ , without loss of generality, assume that  $y$  subdivides a 1-colored edge of  $\overline{H_0}$ . For the graph  $G$ , add  $e$  into  $N_{(0,1)}$ . This revised partition  $\{N_{(0,1)}, N_{(0,2)}, N_{(1,2)}\}$  of  $M$  and the resulting subgraphs  $G(\alpha, \beta)$  satisfy (\*). This contradicts that  $G$  is a counterexample.

Since  $c^{-1}(0) \subseteq H_0$  (each component of  $H - H_0 = H^*$  is mono-colored by 1 or 2), for every edge  $e \in M_{(0,\mu)}$  ( $\mu \in \{1, 2\}$ ), by Claim 1, the edge  $e$  has one endvertex incident with two 0-colored edges of  $H_0$  and its another endvertex belongs to  $V(H - H_0) = V(H^*)$ . That is,

$$M_{(0,0)} = \emptyset, \text{ and } M_{(0,1)} \cup M_{(0,2)} \subseteq E(H_0, H^*).$$

Let

$$G' = \overline{G - M_{(0,1)} - M_{(0,2)}}.$$

Then  $E(G'/F) \subseteq M_{(1,1)} \cup M_{(1,2)} \cup M_{(2,2)}$ .

**Claim 2.** *The graph  $G'/F$  is acyclic.*

Suppose to the contrary that  $G'/F$  contains a circuit  $Q$  (including loops). In the graph  $\overline{G - E(Q)}$ , the spanning subgraph  $H$  retains as a semi-Kotzig frame.

Then the smaller graph  $\overline{G - E(Q)}$  satisfies (\*):  $\overline{H_0}$  has a semi-Kotzig coloring  $c_0$ , and  $M - E(Q)$  has a partition  $\{N_{(0,1)}, N_{(0,2)}, N_{(1,2)}\}$ , and  $H^*$  has a decomposition  $\{H_1^*, H_2^*\}$ . So add all edges of  $E(Q)$  into  $N_{(1,2)}$ . This revised partition  $\{N_{(0,1)}, N_{(0,2)}, N_{(1,2)}\}$  of  $M$  and its resulting subgraphs  $G(\alpha, \beta)$  also satisfy (\*) since  $C_{(1,2)} \cup H^*$  corresponds to an even 2-factor of  $\overline{G(1, 2)} = \overline{G[C_{(1,2)} \cup H^* \cup N_{(1,2)}]}$ . This is a contradiction. So Claim 2 follows.

By Claim 2, each component  $T$  of  $G'/F$  is a tree. Along the tree  $T$ , we can modify the parity 3-edge-coloring  $c$  of  $H$  as follows:

(\*\*) properly switch colors for some circuits in  $F$  so that every edge of  $T$  is incident with four same colored edges.

Note that Rule (\*\*) is feasible by Claim 2 since  $G'/F$  is acyclic. Furthermore, under the modified parity 3-edge-coloring  $c$ ,  $M_{(1,2)} = \emptyset$ . So

$$M = M_{(0,1)} \cup M_{(0,2)} \cup M_{(1,1)} \cup M_{(2,2)}.$$

The colors of all  $H_i$ 's ( $i \geq 1$ ) give a decomposition  $\{H_1^*, H_2^*\}$  of  $H^*$  where  $H_\mu^*$  consists of all circuits of  $H^*$  mono-colored by  $\mu$  for  $\mu = 1$  and 2.

Let

$$G'' = G/H$$

where  $E(G'') = M$ . Then  $G''$  is even since  $H$  is a frame. For a vertex  $w$  of  $G''$  corresponding to a component  $H_i$  with  $i \geq 1$ , there is a  $\mu \in \{1, 2\}$  such that all edges incident with  $w$  belong to  $M_{(0,\mu)} \cup M_{(\mu,\mu)}$ . Define

$$N_{(0,\mu)} = M_{(0,\mu)} \cup M_{(\mu,\mu)}$$

for each  $\mu \in \{1, 2\}$ , and

$$N_{(1,2)} = M_{(1,2)} = \emptyset.$$

Hence, a vertex of  $G''$  corresponding to  $H_i$  with  $i \geq 1$  either has degree in  $G''[N_{(0,\mu)}]$  the same as its degree in  $G''$  or has degree zero (by Rule (\*\*)). So every vertex of  $G''[N_{(0,\mu)}]$  which is different from the vertex corresponding to  $H_0$  has even degree. Since every graph has even number of odd-degree vertices, it follows that  $G''[N_{(0,\mu)}]$  is an even subgraph.

For each  $\mu \in \{1, 2\}$ , let  $G(0, \mu) = N_{(0,\mu)} \cup (c^{-1}(0) \cup c^{-1}(\mu))$ . Since  $G''[N_{(0,\mu)}]$  is an even subgraph of  $G''$ , the even subgraph  $c^{-1}(0) \cup c^{-1}(\mu)$  corresponds to an even 2-factor of  $G(0, \mu)$ . And let  $G(1, 2) = F = c^{-1}(1) \cup c^{-1}(2)$  (here,  $N_{(1,2)} = \emptyset$ ). So  $G$  has the property (\*), a contradiction. This completes the proof of (\*).  $\square$

**Proof of Theorem 1.17.** Let  $G$  be a graph with a semi-Kotzig frame. Then  $G$  satisfies (\*) and therefore is covered by three subgraphs  $G(\alpha, \beta)$  ( $\alpha, \beta \in \mathbb{Z}_3$  and  $\alpha < \beta$ ) as stated in (\*).

Applying Lemma 2.1 to the three graphs  $\overline{G(\alpha, \beta)}$ , each  $G(0, \mu)$  has a 2-even-subgraph cover  $\mathcal{C}_{(0,\mu)}$  which covers the edges of  $C_{(0,\mu)} \cup H_\mu^*$  once and the edges in  $N_{(0,\mu)}$  twice, and  $G(1, 2)$  has a 2-even-subgraph cover  $\mathcal{C}_{(1,2)}$  which covers the edges of  $C_{(1,2)} \cup H^*$  once and the edges in  $N_{(1,2)}$  twice. So  $\bigcup \mathcal{C}_{(\alpha,\beta)}$  is a 6-even-subgraph double cover of  $G$ . This completes the proof.  $\square$

**Remark.** In [5], Häggkvist and Markström proposed another conjecture which strengthens Theorems 1.2, 1.5 and 1.8 as follows.

**Conjecture 2.3** (Häggkvist and Markström, [5]). *If a cubic bridgeless graph contains a connected 3-edge-colorable cubic graph as a spanning minor, then  $G$  has a 6-even-subgraph double cover*

In fact, Conjecture 2.3 is equivalent to that every bridgeless cubic graph has a 6-even-subgraph double cover. It can be shown that the condition in Conjecture 2.3 is true for all cyclically 4-edge-connected cubic graphs.

Consider a cyclically 4-edge-connected cubic graph  $G$  since a smallest counterexample to the 6-even-subgraph double cover problem is cyclically 4-edge-connected and cubic. By the Matching Polytop Theorem of Edmonds [2],  $G$  has a 2-factor  $F$  such that  $G/F$  is 4-edge-connected. By Tutte & Nash-Williams Theorem ([15, 19]),  $G/F$  contains two edge-disjoint spanning trees  $T_1$  and  $T_2$ . By a theorem of Itai and Rodeh ([11]),  $T_1$  contains a parity subgraph  $P$  of  $G/F$ . After suppressing all degree two vertices of  $G - P$ , the graph  $\overline{G - P}$  is 3-edge-colorable and connected since  $G/F - P$  is even and  $T_2 \subset G/F - P$ . So every cyclically 4-edge-connected cubic graph does contain a connected 3-edge-colorable cubic graph as a spanning minor.



## References

- [1] B. Alspach, L. Goddyn and C.-Q. Zhang, Graphs with the circuit cover property, *Trans. Amer. Math. Soc.* **344** (1994) 131-154.
- [2] J. Edmonds, Maximum matching and a polyhedron with  $(0,1)$ -vertices, *J. Res. Nat. Bur. Standards B* **69** (1965) 125-130.
- [3] L.A. Goddyn, *Cycle Covers of Graphs*, Ph.D Thesis, University of Waterloo, 1988.
- [4] L.A. Goddyn, Cycle double covers of graphs with Hamilton paths, *J. Combin. Theory Ser. B* **46** (1989) 253-254.
- [5] R. Häggkvist and K. Markström, Cycle double covers and spanning minors I, *J. Combin. Theory Ser. B* **96** (2006) 183-206.
- [6] R. Häggkvist and K. Markström, Cycle double covers and spanning minors II, *Discrete Math.* **306** (2006) 726-778.
- [7] R. Häggkvist and S. McGuinness, Double covers of cubic graphs with oddness 4, *J. Combin. Theory Ser. B* **93** (2005) 251-277.
- [8] A. Huck, On cycle-double covers of graphs of small oddness, *Discrete Math.* **229** (2001) 125-165.
- [9] A. Huck and M. Kochol, Five cycle double covers of some cubic graphs, *J. Combin. Theory Ser. B* **64** (1995) 119-125.
- [10] A. Hoffmann-Ostenhof, Nowhere-Zero Flows and Structures in Cubic Graphs, PhD thesis.
- [11] A. Itai and M. Rodeh, Covering a graph by circuits, in: *Automata, Languages and Programming, Lecture Notes Comput. Sci.* **62**, Springer-Verlag, Berlin, 1978, pp. 289-299.
- [12] B. Jackson, On circuit covers, circuit decompositions and Euler tours of graphs, in: *Surveys in Combinatorics*, (Keele ed.), *London Math. Soc. Lecture Notes Ser.* **187**, Cambridge Univ. Press, Cambridge, 1993, pp. 191-210.
- [13] F. Jaeger, A survey of the cycle double cover conjecture, in *Cycles in Graphs* (B. Alspach and C. Godsil, eds.), *Ann. Discrete Math.* **27** (1985) pp. 1-12.
- [14] A. Kotzig, Hamilton graphs and Hamilton circuits, in: *Theory of Graphs and its Applications*, Proceedings of the Symposium of Smolenice, 1963, Publ. House Czechoslovak Acad. Sci., Prague, 1964, pp. 63-82.
- [15] C. St. J. A. Nash-Williams, Edge disjoint spanning trees of finite graphs, *J. London Math. Soc.* **36** (1961) 445-450.
- [16] P.D. Seymour, Sums of circuits, in: *Graph Theory and Related Topics* (J.A. Bondy and U.S.R. Murty, eds.), Academic Press, New York, 1979, pp. 342-355.
- [17] G. Szekeres, Polyhedral decompositions of cubic graphs, *Bull. Austral. Math. Soc.* **8** (1973) 367-387.
- [18] M. Tarsi, Semi-duality and the cycle double cover conjecture, *J. Combin. Theory Ser. B* **41** (1986) 332-340.
- [19] W. T. Tutte, On the problem of decomposing a graph into  $n$  connected factors, *J. London Math. Soc.* **36** (1961) 221-230.
- [20] C.-Q. Zhang, *Integer Flows and Cycle Covers of Graphs*, Marcel Dekker, New York, 1997.

- [21] X. Zhang and C.-Q. Zhang, Kotzig frames and circuit double covers, submitted.